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THE EFFECT OF ASYMMETRY ON THE GYRATIONS OF A BALLISTIC MISSILE DESCENDING THROUGH THE ATMOSPHERE

by

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R O Y A L   A I R C R A F T   E S T A B L I S H M E N T

(FARNBOROUGH)

THE EFFECT OF ASYMMETRY ON THE GYRATIONS OF A
BALLISTIC MISSILE DESCENDING THROUGH THE ATMOSPHERE

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SUMMARY

This paper investigates mathematically the effect on the pitch-yaw oscillations of a spinning re-entry body, which will arise as a consequence of the body having a small amount of asymmetry.

It shows how this effect can be related to an equivalent re-entry disturbance for a corresponding symmetrical body.

Some numerical checks, obtained on a digital calculating machine, are included, as substantiation of the mathematical theory.

CONTENTS

ILLUSTRATION - Fig.1 - Response of spinning asymmetric missile, during
re-entry referred to missile axes

DETACHABLE ABSTRACT CARDS

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1   INTRODUCTION

The oscillatory motion of a spinning body, descending through the earth's atmosphere, has been the subject of a fair amount of theoretical research over the last few years. The problem is to derive expressions, preferably reasonably simple ones, which will define the oscillatory motion in terms of two groups of parameters.

(1)   the body's inherent characteristics - mass, moments of inertia, aerodynamic derivatives, etc.,

(2)   the flight conditions at re-entry - velocity, spin rate, the body's orientation, etc.

This is an important issue, because the oscillatory motion of a re-entry body has various practical repercussions. It governs, for example, the magnitude of the lateral loading experienced by the body during descent and this may be a critical design feature. It is also essential to have a theoretical insight into this problem in order to evaluate flight data (e.g. accelerometer recordings) obtained during a re-entry trial, as for example with Black Knight.
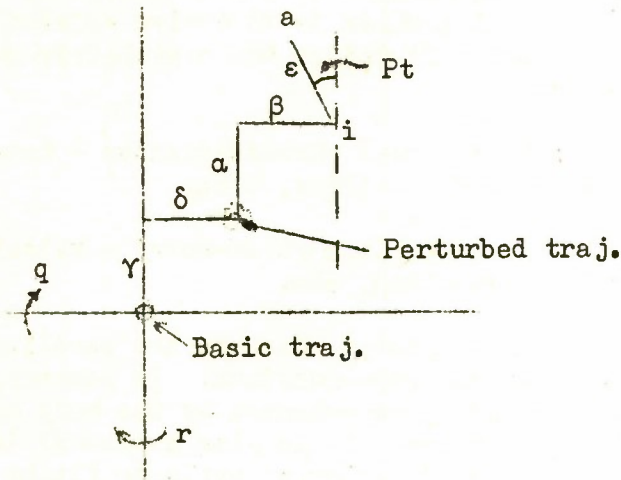
In Ref.1, the re-entry oscillation problem is treated more comprehensively, so it is believed, than elsewhere. There are inevitably, however, various additions and refinements which might in principle be made. One in particular which has occasionally given rise to speculation, is what would happen if the re-entry body were not quite symmetrical. Ref.1 assumes a perfectly symmetrical body which is, of course, normally the designer's intention, but, in practice, there may well be some departure from this. The present paper deals with the problem of calculating the effect on the oscillatory motion which will result from such asymmetry.

It is not difficult to see that this might well be important. Any asymmetry in a spinning body will constitute a forcing input at the spin frequency, this frequency being substantially constant throughout. The natural frequency of the body's oscillations, however, increases steadily during re-entry (until a time shortly before ground level) due to the growth of the dynamic pressure, $\frac{1}{2}\rho V^2$, in the aerodynamic forces. Almost always this natural frequency varies during descent from well below the spin rate to well above it. The system passes, therefore, through a state of resonance. As is well known, if a forcing condition at or near resonance of an oscillating system is maintained, large amplitudes of motion will ensue. The general issue which arises in the present problem is to discover how the response of a forced oscillatory system is modified by the continuous state of flux of some of the system's parameters.

2   MATHEMATICAL DEFINITION OF THE PROBLEM

Asymmetry in a re-entry vehicle can take different forms (c.f. Ref.2). We are concerned here with aerodynamic asymmetry, that is to say a case in which the roll inertial axis is inclined at a small angle (ε) to the aerodynamic axis. The moments of inertia in pitch and yaw are equal.

We follow the same approach as in Ref.1, so that, as seen from in front
with small perturbations we have the situation:-

a   aerodynamic axis

i   inertial axis

a, β components of incidence of inertial axis

γ, β angular perturbation components of the trajectory

q, r angular rate components of inertial axis

ε angle between a and i

A   roll moment of inertia

B   pitch (or yaw) moment of inertia

m   mass

P   spin rate

The equations of angular motion of the inertial axis are:-

$$m \left[\overline{q + ir} - \overline{\dot{a - i\beta}} \right] = \tfrac{1}{2}\rho VSC_{L_a} \left[\overline{a - i\beta} + \varepsilon e^{iPt} \right] \qquad (1)$$

$$B \, \overline{\dot{q + ir}} - i \, AP \, \overline{q + ir} = \tfrac{1}{2}\rho V^2 S\ell \left[C_{m_a} \left(\overline{a - i\beta} + \varepsilon e^{iPt} \right) + C_{m_q} \left(\overline{q + ir} + i\varepsilon P e^{iPt} \right) \frac{\ell}{V} \right].$$

$$\dots (2)$$

These equations correspond to (7) and (8) in Ref.1, with the additional terms in the aerodynamic forces arising from the asymmetry. We now eliminate $q + ir$, to obtain a differential equation in $\alpha - i\beta$.

As is brought out in Ref.1, various terms appear which are quite negligible in practice. Retaining only terms of numerical significance, we obtain:-

$$\overline{\ddot{\alpha - i\beta}} + \overline{\dot{\alpha - i\beta}}\left[\tfrac{1}{2}\rho VS\left(\frac{C_{L_\alpha}}{m} - \frac{\ell^2 C_{m_q}}{B}\right) - i\,\frac{AP}{B}\right] + \overline{\alpha - i\beta}\left[-\frac{\tfrac{1}{2}\rho V^2 S\ell C_{m_\alpha}}{B} - i\,\frac{AP}{B}\,\frac{\tfrac{1}{2}\rho VS C_{L_\alpha}}{m}\right]$$

$$= \frac{\tfrac{1}{2}\rho V^2 S\ell C_{m_\alpha}}{B}\,\varepsilon\,e^{iPt}\ . \tag{3}$$

This is the counterpart of equation (12) in Ref.1, the asymmetry appearing on the right-hand side of the equation.

During a re-entry descent, the velocity is substantially constant until the lower regions of the atmosphere are reached, whilst the air density builds up exponentially with loss of height. For much of the descent, therefore, ρ takes the form

$$\rho = \rho_o\,e^{Kt} \qquad (\rho_o,\ K \text{ are constants})\ .$$

Restricting the investigation to this case, in equation (3) we replace ρ by $\rho_o\,e^{Kt}$

$$\omega^2 = \frac{-\tfrac{1}{2}\rho V^2 S\ell C_{m_\alpha}}{B}\ \text{by}\ \omega_o^2\,e^{Kt}$$

$$a = \tfrac{1}{2}\rho VS\left(\frac{C_{L_\alpha}}{m} - \frac{\ell^2 C_{m_q}}{B}\right)\ \text{by}\ a_o\,e^{Kt}$$

$$b = \tfrac{1}{2}\rho VS\left(\frac{C_{L_\alpha}}{m} + \frac{\ell^2 C_{m_q}}{B}\right)\ \text{by}\ b_o\,e^{Kt}\ .$$

We then arrive at the differential equation

$$\overline{\overset{..}{\alpha-i\beta}} + \overline{\overset{.}{\alpha-i\beta}}\left[a_o e^{Kt} - i\,\frac{AP}{B}\right] + \overline{\alpha-i\beta}\left[\omega_o^2 e^{Kt} - i\,\frac{AP}{2B}(a_o + b_o)e^{Kt}\right] = -\varepsilon\omega_o^2 e^{Kt} e^{iPt}.$$

$$\dots (4)$$

This is the equation whose solution is required.

3 MATHEMATICAL SOLUTION OF THE PROBLEM

We make a change of variables

$$T = e^{\frac{Kt}{2}}, \quad \alpha - i\beta = T^{i\nu}\,e^{-\frac{a_o T^2}{2K}}\,\eta, \quad \text{where} \quad \nu = \frac{AP}{BK}$$

and we write

$$\lambda = \frac{P}{K}.$$

Then equation (4) transforms, mathematically into

$$T^2\frac{d^2\eta}{dT^2} + T\frac{d\eta}{dT} + \left[\frac{-4\omega_o^2 T^2}{K^2} - \frac{2a_o T^2}{K^2} - \frac{2ib_o}{K^2}\,\frac{AP}{B}T^2 - \frac{a_o^2 T^4}{K^2} - (i\nu)^2\right]\eta$$

$$= -\varepsilon\,\frac{4\omega_o^2}{K^2}\,T^{2+i(2\lambda-\nu)}\,e^{\frac{a_o T^2}{2K}}. \quad (5)$$

3.1 Special case of no damping or lift

It will simplify the mathematical development of the solution of the general equation if we consider first the special case in which there is neither aerodynamic damping nor lift, namely the case in which the aerodynamic forces on the body consist solely of a restoring moment. In this case, a_o and b_o are both zero and equation (5) becomes

$$T^2\frac{d^2\eta}{dT^2} + T\frac{d\eta}{dT} + \left[\frac{-4\omega_o^2 T^2}{K^2} - (i\nu)^2\right]\eta = -\varepsilon\,\frac{4\omega_o^2}{K^2}\,T^{2+i(2\lambda-\nu)}. \quad (6)$$

Now make the further substitution

$$\frac{2\omega_0 T}{K} = z$$

in which case the equation becomes

$$z^2 \frac{d^2\eta}{dz^2} + z \frac{d\eta}{dz} + \left[z^2 - (i\nu)^2\right]\eta = -\varepsilon\left(\frac{2\omega_0}{K}\right)^{-i(2\lambda-\nu)} z^{2+i(2\lambda-\nu)}. \qquad (7)$$

If this differential oquation had zero on the right-hand side, it would, of course, be of the standard Bessel's equation form, of order $i\nu$. As it is, it can be identified as one of a number which have a close affinity with Bessel's form and which, at some time, have received attention by mathematicians. Equation (7) is of the form involving Lommel's functions. (page 40, Vol.2 Ref.3, or page 345, Ref.4.)

The complete solution of oquation (7) will inevitably involve arbitrary amounts of the functions $J_{i\nu}(z)$, $J_{-i\nu}(z)$. However, let us look at a particular solution, (equation (69), page 40, Vol.2, Ref.3)

$$\eta = -\varepsilon\left(\frac{2\omega_0}{K}\right)^{-2i\lambda+i\nu} S_{1+2i\lambda-i\nu,\ i\nu}(z). \qquad (8)$$

$S_{1+2i\lambda-i\nu,\ i\nu}(z)$ is a Lommel function, z being the current variable. The other quantities, which are constants, play a roll akin to the order in normal Bessel's functions.

Lommel's functions (of this kind) are not tabulated anywhere, as far as the present writer can discover, so that one cannot proceed by consulting tables. However, some of their mathematical properties have been explored, including how they behave for either small or large values of the argument, z, which properties are particularly useful in the present context.

By taking the same functional form of the solution (8) and considering in turn its behaviour for small (z), large altitude and large (z), small altitude, we can link the initial re-entry motion at very high altitude including a determination of the arbitrary pre-entry motion, with the final stages of re-entry motion at low altitude though being unable to analyse mathematically the motion in between. We will therefore evaluate the two cases of low and high z in turn.

3.1.1 Small values of z (large altitude)

Let us look first at small values of z, corresponding to the initial stages of the re-entry problem when the altitude is large.

The 's' function can then be expanded in ascending powers of z.

Equation (8) then leads to

$$\eta = -\varepsilon \left(\frac{2\omega_o}{K}\right)^{-2i\lambda+i\nu} \left[\frac{z^{2+2i\lambda-i\nu}}{2^2(1+i\lambda-i\nu)(1+i\lambda)} - \frac{z^{4+2i\lambda-i\nu}}{2^4(1+i\lambda-i\nu)(1+i\lambda)(2+i\lambda-i\lambda)(2+i\lambda)} + \cdots \right]$$

If we now write $\omega_o T = \omega$, this reduces to

$$\alpha - i\beta = -\varepsilon T^{2i\lambda} \left[\frac{\left(\frac{\omega}{K}\right)^2}{(1+i\lambda-i\nu)(1+i\lambda)} - \frac{\left(\frac{\omega}{K}\right)^4}{(1+i\lambda-i\nu)(1+i\lambda)(2+i\lambda-i\nu)(2+i\lambda)} + \cdots \right] \cdot$$

$$\cdots (9)$$

The term $T^{2i\lambda} = T^{\frac{2iP}{K}} = e^{iPt}$ represents simply a rotation at the spin rate, P. It means that equation (9), with this term removed, gives the motion referred to missile-fixed axes.

Employing a suffix, m, to indicate reference to missile axes, we have

$$\alpha_m - i\beta_m = -\varepsilon \left[\frac{\left(\frac{\omega}{K}\right)^2}{(1+i\lambda-i\nu)(1+i\lambda)} - \frac{\left(\frac{\omega}{K}\right)^4}{(1+i\lambda-i\nu)(1+i\lambda)(2+i\lambda-i\nu)(2+i\lambda)} + \cdots \right] \quad (10)$$

ω, which is the natural frequency of the missile in pitch under aerodynamic forces, gradually increases from zero during re-entry. Hence, the solution we have arbitrarily chosen represents the case in which the missile's inertial axis is lined up, initially, with the flight direction, and commences its movement away under the effect of the growing, asymmetric, aerodynamics.

Equation (10) shows this in a typical case, to be of the following form:-
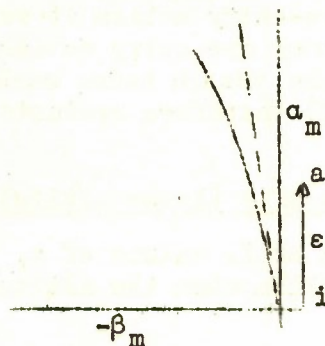
The first term in the bracket dominates initially. In the most common case, λ is much larger than either 1 or ν.

In these circumstances,

$$\alpha_m - i\beta_m \simeq - \varepsilon \frac{\left(\frac{\omega}{K}\right)^2}{-\lambda^2 + 2i\lambda} \simeq \frac{\varepsilon}{\lambda^2}\left(\frac{\omega}{K}\right)^2\left(1 + \frac{2i}{\lambda}\right) = \varepsilon\frac{\omega^2}{P^2}\left(1 + \frac{2i}{\lambda}\right)$$

showing that the response, on missile axes, starts off (for large λ) at an angle $2/\lambda$ to the vector defining the asymmetry.

3.1.2 Large values of z (low altitude)

Now we turn to the situation when z is large, corresponding to the later stages of re-entry, when the altitude is much reduced.

We are dealing of course with the same solution, as defined by equation (8). Now we wish to expand in descending powers of z.

To do this, we note that (equation 7, page 40, Ref.3)

$$S_{1+2i\lambda-i\nu,\,i\nu}(z) = S_{1+2i\lambda-i\nu,\,i\nu}(z)$$

$$-2^{2i\lambda-i\nu}\frac{\Gamma(1+i\lambda)\Gamma(1+i\lambda-i\nu)}{\sin i\pi\nu}\left[\cos\frac{\pi}{2}(1+2i\lambda-2i\nu)J_{-i\nu}(z)\right.$$

$$\left. - \cos\frac{\pi}{2}(1+2i\lambda)J_{i\nu}(z)\right] \quad (11)$$

'S' is an alternative form of Lommel function, readily expandable in descending powers. Γ is the gamma function, J Bessel's function.

For large z,

$$S_{1+2i\lambda-i\nu,\,i\nu}(z) \sim z^{2i\lambda-i\nu}\left[1 - \frac{2i\lambda}{z^2}\frac{2i(\lambda-\nu)}{} + \dots\right]$$

$$J_{i\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos\left(z - \frac{\pi}{4} - \frac{i\pi\nu}{2}\right) = \frac{1}{(2\pi z)^{\frac{1}{2}}}\left[e^{iz - \frac{i\pi}{4} + \frac{\pi\nu}{2}} + e^{-iz + \frac{i\pi}{4} - \frac{\pi\nu}{2}}\right]$$

$$J_{-i\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos\left(z - \frac{\pi}{4} + \frac{i\pi\nu}{2}\right) = \frac{1}{(2\pi z)^{\frac{1}{2}}}\left[e^{iz - \frac{i\pi}{4} - \frac{\pi\nu}{2}} + e^{-iz + \frac{i\pi}{4} + \frac{\pi\nu}{2}}\right].$$

When these expressions are substituted into (11), we obtain, after a little algebraic simplification,

$$s_{1+2i\lambda-i\nu,\,i\nu}(z) \sim z^{2i\lambda-i\nu}\left[1 + \frac{4\lambda(\lambda-\nu)}{z^2}\right] +$$

$$+ \frac{i2^{2i\lambda-i\nu}}{(2\pi z)^{\frac{1}{2}}}\Gamma(1+i\lambda)\Gamma(1+i\lambda-i\nu)\left[e^{-iz+\frac{i\pi}{4}}e^{\pi\left(\lambda-\frac{\nu}{2}\right)} + e^{-iz+\frac{3i\pi}{4}}e^{-\pi\left(\lambda-\frac{\nu}{2}\right)}\right]$$

$$\dots (12)$$

Thus from equation (8), and remembering that

$$\alpha - i\beta = T^{i\nu}\eta\,, \quad z = \frac{2\omega}{K} \quad \text{etc.,} \quad \text{we obtain}$$

$$\alpha - i\beta \sim -\varepsilon T^{2i\lambda}\left[1 + \frac{P^2}{\omega^2}\left(1 - \frac{A}{B}\right)\right] -$$

$$- \frac{i\varepsilon}{2\sqrt{\pi}}\left(\frac{K}{\omega}\right)^{\frac{1}{2}}\left(\frac{K}{\omega_0}\right)^{2i\lambda-i\nu}\Gamma(1+i\lambda)\Gamma(1+i\lambda-i\nu)\left[e^{\pi\left(\lambda-\frac{\nu}{2}\right)}e^{i\left(\frac{APt}{2B}+\frac{2\omega}{K}+\frac{\pi}{4}\right)}\right.$$

$$\left. + e^{-\pi\left(\lambda-\frac{\nu}{2}\right)}e^{i\left(\frac{APt}{2B}-\frac{2\omega}{K}+\frac{3\pi}{4}\right)}\right].$$

$$\dots (13)$$

Equation (13) represents the situation which ensues at low altitudes. The first term represents a vector which is locked to missile axes, and rapidly shrinking onto $-\varepsilon$. The remainder represents two vectors, rotating at rates $\frac{AP}{2B} \pm \omega$ in relation to space axes, and shrinking in inverse proportion to $\sqrt{\omega}$.

In general this is as far as the mathematical development can be taken, and tables of gamma functions can be consulted for numerical values. However, it frequently happens that λ is much larger than 1 (or ν) and in that case approximations to the gamma functions can be made.

The standard approximation is:-

for large X, $\qquad \log_e \Gamma(X) \sim (X - \tfrac{1}{2})\log_e X - X + \tfrac{1}{2}\log_e 2\pi$.

Thus, $\qquad \Gamma(1 + i\lambda) \sim \sqrt{2\pi\lambda}\; e^{-\frac{\pi\lambda}{2} + i\left[\lambda(\log\lambda - 1) - \frac{1}{2\lambda} + \frac{\pi}{4}\right]}$

$$\Gamma(1+i\lambda-i\nu) \sim \sqrt{2\pi(\lambda-\nu)}\; e^{-\frac{\pi}{2}(\lambda-\nu) + i\left[(\lambda-\nu)\{\log(\lambda-\nu) - 1\} - \frac{1}{2(\lambda-\nu)} + \frac{\pi}{4}\right]} .$$

On substituting these expressions into equation (13) we obtain, after a little simplification,

$$\alpha - i\beta \sim -\varepsilon\left[1 + \frac{P^2}{\omega^2}\left(1 - \frac{A}{B}\right)\right]e^{iPt} + \varepsilon\sqrt{\frac{\pi K \lambda(\lambda-\nu)}{\omega}}\left[e^{i\left(\frac{2\omega}{K} + \phi\right)} + e^{-\pi(2\lambda-\nu) + i\left(-\frac{2\omega}{K} + \frac{\pi}{2} + \phi\right)}\right]e^{iPt}$$

where

$$\phi = \lambda \log 2\lambda + (\lambda-\nu)\log 2(\lambda-\nu) - 2\lambda + \nu - \frac{1}{2\lambda} - \frac{1}{2(\lambda-\nu)} + \frac{\pi}{4} - (2\lambda-\nu)\log\left(\frac{2\omega}{K}\right).$$

$$\dots (14)$$

The equations (14) define the orientation of the inertial axis at low altitudes for large λ (say >10), for the case in which this axis had zero displacement from the flight direction initially.

These expressions refer to space axes. Deletion of the terms e^{iPt} gives $\alpha_m - i\beta_m$, the incidence in relation to missile axes.

3.2 <u>General case including damping and lift</u>

We return now to the more general case in which damping and lift terms are retained, the appropriate differential equation being equation (5), and seek a solution along similar lines to the classical mathematics of the above special case.

In the η term in that equation, $\dfrac{2a_o T^2}{K^2}$ will always, in practice, be very small in relation to $\dfrac{4\omega_o^2 T^2}{K^2}$, and will therefore be neglected.

We proceed with

$$T^2 \frac{d^2\eta}{dT^2} + T \frac{d\eta}{dT} + \left[\left(\frac{4\omega_o^2}{K^2} - \frac{2ib_o}{K^2} \frac{AP}{B} \right) T^2 - \frac{a_o^2 T^4}{K^2} - (i\nu)^2 \right] \eta$$

$$= -\varepsilon \frac{4\omega_o^2}{K^2} T^{2+i(2\lambda-\nu)} e^{\frac{a_o T^2}{2K}} \quad (15)$$

and make the substitution,

$$\left(\frac{4\omega_o^2}{K^2} - \frac{2ib_o}{K^2} \frac{AP}{B} \right) T^2 = z^2 .$$

We then obtain

$$z^2 \frac{d^2\eta}{dz^2} + z \frac{d\eta}{dz} + \left[z^2 - \frac{a_o^2 K^2 z^4}{\left(4\omega_o^2 - 2ib_o \frac{AP}{B} \right)^2} - (i\nu)^2 \right] \eta$$

$$= -\varepsilon \left(\frac{2\omega_o}{K} \right)^{-i(2\lambda-\nu)} z^{2+i(2\lambda-\nu)} e^{\frac{a_o K}{2\left(4\omega_o^2 - 2ib_o \frac{AP}{B} \right)} z^2} \quad (16)$$

to a sufficient accuracy.

If we let $c = \dfrac{a_o K}{2\left(4\omega_o^2 - 2ib_o \frac{AP}{B} \right)}$ this equation becomes,

$$z^2 \frac{d^2\eta}{dz^2} + z \frac{d\eta}{dz} + \left[z^2 - 4c^2 z^4 - (i\nu)^2 \right] \eta = -\varepsilon \left(\frac{2\omega_o}{K} \right)^{-i(2\lambda-\nu)} z^{2+i(2\lambda-\nu)} e^{cz^2} .$$

$$\dots \quad (17)$$

This is the counterpart to equation (7) in Section 3.1. The term $4c^2 z^4$ in the coefficient of η is negligible, in a typical re-entry case, in comparison with z^2. It is best retained, however, since it assists the mathematical development.

3.2.1 Ascending powers of z

Following the lines of Section 3.1, we seek first a solution in ascending powers of z corresponding to Lommel's function, s. Denoting the corresponding function by $\bar{s}$, and writing $\mu = 1 + i(2\lambda-\nu)$, we can devise

$$\bar{s} = e^{cz^2} \left[a_1 z^{\mu+1} + a_3 z^{\mu+3} + a_5 z^{\mu+5} + \dots \right]$$

in which

$$a_1 = \frac{1}{(\mu+1)^2 - (i\nu)^2} ; \qquad a_3 = - \frac{1 + 4c(\mu+2)}{\left[(\mu+1)^2 - (i\nu)^2 \right] \left[(\mu+3)^2 - (i\nu)^2 \right]} ;$$

$$a_5 = \frac{\left[1 + 4c(\mu+2) \right] \left[1 + 4c(\mu+4) \right]}{\left[(\mu+1)^2 - (i\nu)^2 \right] \left[(\mu+3)^2 - (i\nu)^2 \right] \left[(\mu+5)^2 - (i\nu)^2 \right]} ; \quad \text{etc.}$$

Thus

$$\eta = -\varepsilon \left(\frac{2\omega_o}{K} \right)^{-i(2\lambda-\nu)} \bar{s}_{1+2i\lambda-i\nu, i\nu}(z) \qquad (18)$$

corresponding to the case in which the inertial axis is lined up initially with the flight direction.

3.2.2 Descending powers of z

Here we seek a form of solution to equation (17) corresponding to Lommel's function S. Denoting this by $\bar{S}$, we can obtain

$$\bar{S} = e^{cz^2} z^{\mu-1} \left[a_o + \frac{a_2}{z^2} + \frac{a_4}{z^4} + \cdots \right]$$

in which

$$a_o = \frac{1}{1+4c\mu} \; ; \quad a_2 = - \frac{[(\mu-1)^2 - (i\nu)^2]}{[1+4c\mu][1+4c(\mu-2)]}$$

$$a_4 = \frac{[(\mu-1)^2 - (i\nu)^2][(\mu-3)^2 - (i\nu)^2]}{[1+4c\mu][1+4c(\mu-2)][1+4c(\mu-4)]} \; ; \quad \text{etc.}$$

Thus, $\qquad \eta = -\varepsilon \left(\frac{2\omega_o}{K} \right)^{-i(2\lambda-\nu)} \bar{S}_{1+2i\lambda-i\nu,\,i\nu}(z)$

providing a convenient form of solution for large z.

3.2.3 Complementary function

We require a suitable form for the complementary function of equation (17), i.e. the solution when the right-hand side is set to zero, and explore the form

$$\eta = e^{cz^2} \left[a_o z^r + a_2 z^{r+2} + a_4 z^{r+4} + \cdots \right].$$

By substitution in (17), it is found that

$r = +i\nu$	or $r = -i\nu$
a_o arbitrary	a_o arbitrary

$$a_2 = - \frac{a_o[1 + 4c(i\nu + 1)]}{(i\nu+2)^2 - (i\nu)^2} \qquad\qquad a_2 = - \frac{a_o[1 + 4c(-i\nu+1)]}{(-i\nu+2)^2 - (-i\nu)^2}$$

$$a_4 = \frac{a_o[1 + 4c(i\nu+1)][1 + 4c(i\nu+3)]}{[(i\nu+2)^2 - (i\nu)^2][(i\nu+4)^2 - (i\nu)^2]} \qquad a_4 = \frac{a_o[1 + 4c(-i\nu+1)][1 + 4c(-i\nu+3)]}{[(-i\nu+2)^2 - (-i\nu)^2][(-i\nu+4)^2 - (-i\nu)^2]}$$

$$\text{etc.} \qquad\qquad\qquad\qquad\qquad \text{etc.}$$

Thus, the two solutions of this form can be written

$$A_1 \; e^{cz^2} \; z^{i\nu} \left\{ 1 - \frac{[1 + 4c(i\nu+1)]z^2}{2^2(i\nu+1)\,\lfloor 1} + \frac{[1 + 4c(i\nu+1)][1 + 4c(i\nu+3)]z^4}{2^4(i\nu+1)(i\nu+2)\,\lfloor 2} - \dots \right\} \quad (19)$$

$$A_2 \; e^{cz^2} \; z^{-i\nu} \left\{ 1 - \frac{[1 + 4c(-i\nu+1)]z^2}{2^2(-i\nu+1)\,\lfloor 1} + \frac{[1 + 4c(-i\nu+1)][1 + 4c(-i\nu+3)]z^4}{2^4(-i\nu+1)(-i\nu+2)\,\lfloor 2} - \dots \right\} \quad (20)$$

where A_1 and A_2 are arbitrary constants.

As in the standard Bessel case (corresponding to $c = 0$), we take

$$A_1 = \frac{1}{2^{i\nu}\, \Gamma(i\nu+1)} \;, \quad A_2 = \frac{1}{2^{-i\nu}\, \Gamma(-i\nu+1)} \; . \quad \text{(cf. Ref.4,p.40)}$$

Now, in re-entry applications the quantity c is quite small, and also $c^2 z^4$ is always quite small in relation to z^2. Thus for the range of z in which we are interested, equation (17) with the right-hand side set to zero is sensibly equivalent to the standard Bessel form of order $i\nu$, and the complementary function solutions found can be identified for our purpose as $J_{i\nu}(z)$ and $J_{-i\nu}(z)$ respectively.

3.2.4 Complete solution

The solution in which we are interested is as given by equation (18) since this corresponds to zero initial displacement. We require now to express $\bar{s}$ in terms of $\bar{S}$ and the complementary function $J_{i\nu}$ and $J_{-i\nu}$. This will be the counterpart in the more general case to equation (11) in the special case, and will enable an asymptotic value to the solution to be found. This in turn will be used to explore what happens to the motion in the later stages of re-entry.

It is a matter of algebra to show that there is an analogous relation to equation (11), viz:-

$$\bar{s}_{1+2i\lambda-i\nu,\,i\nu}(z) = \bar{S}_{1+2i\lambda-i\nu,\,i\nu}(z)$$

$$- \frac{2^{2i\lambda-i\nu}}{\sin i\pi\nu} \frac{\Gamma(1+i\lambda)\Gamma(1+i\lambda-i\nu)}{\left[\left(\frac{1}{8c}+\frac{3}{2}+i\lambda-\frac{i\nu}{2}\right)\right]} \left[\frac{\Gamma\left(\frac{1}{8c}+\frac{1}{2}-\frac{i\nu}{2}\right)}{(8c)^{1+i\lambda}} \cos \frac{\pi}{2}(1+2i\lambda-2i\nu) J_{-i\nu}(z) \right.$$

$$\left. - \frac{\Gamma\left(\frac{1}{8c}+\frac{1}{2}+\frac{i\nu}{2}\right)}{(8c)^{1+i\lambda-i\nu}} \cos \frac{\pi}{2}(1+2i\lambda) J_{i\nu}(z) \right] \quad (21)$$

where expression (19) and (20) are used for $J_{i\nu}(z)$, $J_{-i\nu}(z)$ in this case.

As in Section 3.1.2, in order to proceed further with the mathematical development it is necessary to approximate to the gamma functions. This is quite feasible, since in practice, 'c' is quite small and $1/c$ quite large.

Using the standard approximation for the Γ-function of a large quantity, it is straightforward to show that for Y large, and much larger than z

$$\Gamma(Y+iz) \sim Y^Y \sqrt{\frac{2\pi}{Y}} e^{-Y - \frac{s^2}{2Y} + iz\left(\log z - \frac{1}{2Y}\right)} .$$

By making use of this formula, it can be demonstrated that

$$\frac{\Gamma\left(\frac{1}{8c}+\frac{1}{2}-\frac{i\nu}{2}\right)}{\Gamma\left(\frac{1}{8c}+\frac{3}{2}+i\lambda-\frac{i\nu}{2}\right)} \frac{1}{(8c)^{1+i\lambda}} = \frac{\Gamma\left(\frac{1}{8c}+\frac{1}{2}+\frac{i\nu}{2}\right)}{\Gamma\left(\frac{1}{8c}+\frac{3}{2}+i\lambda-\frac{i\nu}{2}\right)} \frac{1}{(8c)^{1+i\lambda-i\nu}}$$

$$\sim e^{\frac{a_oK}{2\omega_o}[\lambda(\lambda-\nu)+i(-2\lambda+\nu)]} .$$

Also from Section 3.2.2, we have, for large z,

$$\bar{S}_{1+2i\lambda-i\nu,\,i\nu}(z) \sim e^{cz^2} z^{\mu-1} \left[\frac{1}{1+4c\mu} - \frac{[(\mu-1)^2-(i\nu)^2]}{[1+4c\mu][1+4c(\mu-2)]} \frac{1}{z^2} + 0\left(\frac{1}{z^4}\right) \right]$$

$$\sim e^{cz^2} z^{i(2\lambda-\nu)} \left[1 - \frac{a_oK}{2\omega_o^2} - \frac{ia_oK}{2\omega_o^2}\left(\lambda-\frac{\nu}{2}\right) + \frac{4\lambda(\lambda-\nu)}{z^2} \right] .$$

Hence, [c.f. equation (12), Section 3.1.2]

$$\bar{s}_{1+2i\lambda-i\nu,\,i\nu}(z) \sim e^{cz^2} z^{i(2\lambda-\nu)} \left[1 - \frac{a_0 K}{2\omega_0^2} - \frac{ia_0 K}{2\omega_0^2}\left(\lambda - \frac{\nu}{2}\right) + \frac{4\lambda(\lambda-\nu)}{z^2} \right]$$

$$+ \frac{i2^{i(2\lambda-\nu)}}{(2\pi z)^{\frac{1}{2}}} \Gamma(1+i\lambda)\Gamma(1+i\lambda-i\nu)\, e^{\frac{a_0 K}{2\omega_0^2}[\lambda(\lambda-\nu)+i(-2\lambda+\nu)]} \times$$

$$\times \left[e^{iz+\frac{i\pi}{4}}\, e^{\pi\left(\lambda-\frac{\nu}{2}\right)} + e^{-iz+\frac{3i\pi}{4}}\, e^{-\pi\left(\lambda-\frac{\nu}{2}\right)} \right].$$

$$\ldots (22)$$

Recalling that

$$\eta = -\varepsilon \left(\frac{2\omega_0}{K}\right)^{-i(2\lambda-\nu)} \bar{s}\,,$$

$$\alpha - i\beta = T^{i\nu}\, e^{-\frac{a_0 T^2}{2K}}\, \eta\,,$$

$$K^2 z^2 = \left(4\omega_0^2 - 2ib_0 \frac{AP}{B}\right) T^2$$

and using approximations to the functions $\Gamma(1+i\lambda)$, $\Gamma(1+i\lambda-i\nu)$ as in Section 3.1.2 we can find the asympotic value of $\alpha-i\beta$ corresponding to late stages of re-entry.

This comes out to be

$$\alpha-i\beta \sim -\varepsilon\left[1 + \frac{K}{2}\left\{ -\frac{a_0}{\omega_0^2} + \frac{b_0}{\omega_0^2}\nu\left(\lambda-\frac{\nu}{2}\right) - i\frac{a_0}{\omega_0^2}\left(\lambda-\frac{\nu}{2}\right) \right\} + \frac{P^2}{\omega_0^2}\left(1 - \frac{A}{B}\right) \right]e^{iPt}$$

$$+ \varepsilon\sqrt{\frac{\pi K\lambda(\lambda-\nu)}{\omega}}\, e^{\frac{a_0 K\lambda(\lambda-\nu)}{2\omega_0^2}}\, e^{-\frac{a}{2K}}\left[e^{\frac{b\nu}{2\omega}+i\left(\frac{2\omega}{K}+\phi\right)} \right.$$

$$\left. + e^{-\pi(2\lambda-\nu)-\frac{b\nu}{2\omega}+i\left(-\frac{2\omega}{K}+\frac{\pi}{2}+\phi\right)} \right]e^{iPt}$$

$$\ldots (23)$$

where $\quad \phi = \lambda \log 2\lambda + (\lambda-\nu) \log 2(\lambda-\nu) - 2\lambda + \nu - \dfrac{1}{2\lambda} - \dfrac{1}{2(\lambda-\nu)} + \dfrac{\pi}{4}$

$$- (2\lambda-\nu) \log\left(\dfrac{2\omega}{K}\right) + \dfrac{a_o K}{2\omega_o^2} (-2\lambda+\nu) + \dfrac{b_o K\nu}{8\omega_o^2} \,. \qquad (24)$$

This is the mathematical solution of the problem, subject to certain restrictions on the size of some of the quantities involved (see Section 4). It is the more general form of equation (14), the effect of damping and lift introducing the extra terms in a, b, a_o, b_o.

4   PRACTICAL IMPLICATIONS

We have found [equations (23), (24)] the angular motion of an asymmetric spinning body, occurring at low altitudes, as a consequence of its inherent asymmetry. As appears later in this paragraph, this is the most suitable form for an appraisal of the practical implications of asymmetry. For equations (23) and (24) to be valid, two conditions must be satisfied:-

(a)   damping and lift must be relatively small, i.e. $\dfrac{a_o}{\omega_o^2}$, $\dfrac{b_o}{\omega_o^2}$ must be

small, in comparison with unity,

(b) $\lambda\left(\dfrac{P}{K}\right)$ must be large in comparison with unity.

For all typical re-entry cases, the former condition, (a), is well satisfied. For condition (b), it is sufficient if λ is greater than about 10, and this is frequently the case in re-entry applications. If it should happen that λ is much less than this, it would be necessary to go back to equation (22) and evaluate the Γ-functions from suitable tables.

The effect of asymmetry, as per equation (23), consists, in the later stages of re-entry, of two parts

(1)   a steady displacement of the inertial axis of amount

$$- \varepsilon\left[1 + \dfrac{K}{2}\left\{ -\dfrac{a_o}{\omega_o^2} + \dfrac{b_o}{\omega_o^2}\nu\left(\lambda - \dfrac{\nu}{2}\right) - i\,\dfrac{a_o}{\omega_o^2}\left(\lambda - \dfrac{\nu}{2}\right) \right\} \right] ,$$

in relation to missile axes.

This is approximately equal and opposite to the aerodynamic asymmetry.

(2)   a motion consisting of two rotating vectors of exactly the same type as the normal symmetrical missile undergoes.

If ε is small, as is supposed, then constituent (1) is also small. Interest, in practice, centres on constituent (2). Of this the component vector which rotates in the same sense as the missile spins will be much larger than the counter rotating vector, since the latter involves the factor $e^{-\pi(2\lambda-\nu)}$.

The feature of principal interest, therefore, is the vector

$$\varepsilon \sqrt{\frac{\pi K\lambda(\lambda-\nu)}{\omega}} \, e^{\dfrac{a_o K\lambda(\lambda-\nu)}{2\omega_o^2}} \, e^{-\frac{a}{2K} + \frac{b\nu}{2\omega} + i\left(\frac{2\omega}{K} + \phi\right)}.$$

Now, it is shown in Ref.1, that if a symmetrical missile has a simple precessional motion of magnitude S_E, prior to re-entry, then its motion during re-entry will be given by

$$S_E \sqrt{\frac{AP}{2B\omega}} \, e^{-\frac{a}{2K} + \frac{b\nu}{2\omega} + i\left(\frac{2\omega}{K} + \phi\right)}.$$

These two expressions are of the same form. Hence, by equating them, we can express the effect of asymmetry in terms of an equivalent motion, prior to re-entry for a symmetrical missile

$$S_E \sqrt{\frac{AP}{2B}} = \varepsilon \, e^{\dfrac{a_o K\lambda(\lambda-\nu)}{2\omega_o^2}} \sqrt{\pi K\lambda(\lambda-\nu)}$$

Therefore

$$\frac{S_E}{\varepsilon} = e^{\dfrac{a_o K\lambda(\lambda-\nu)}{2\omega_o^2}} \sqrt{\frac{2\pi\lambda(\lambda-\nu)}{\nu}}. \qquad (25)$$

For this relationship to be valid, the further condition that ν must not be less than about unity must be satisfied. This is usually so in practice.

Rewritten in terms of basic design parameters equation (25) becomes

$$\frac{S_E}{\varepsilon} = e^{-\dfrac{\ell P^2(\sigma^2 C_{L_\alpha} - C_{m_q})(1 - A/B)}{2KV_E C_{m_\alpha}}} \sqrt{2\pi \frac{P}{K}\left(\frac{B}{A} - 1\right)} \qquad (26)$$

where

C_{L_α}, C_{m_q}, C_{m_α} = aerodynamic derivatives of lift, pitch damping, and static pitching moment

ℓ = reference length

σ = non-dimensional radius of gyration in pitch, i.e. ratio of pitch radius of gyration to reference length

P = spin rate

A, B = moments of inertia of re-entry missile in roll, pitch

V_E = re-entry velocity

K = given by air density variation, $\rho = \rho_o e^{Kt}$

ε = asymmetry in missile

S_E = equivalent re-entry perturbation for symmetrical missile

Equation (26) is the principal practical result of the present enquiry. It defines the situation in which the missile finds itself after passing through resonance, and expresses it in terms of the equivalent re-entry disturbance for a symmetrical body.

It is interesting to note that the presence of damping (and lift) in the re-entry missile increases the magnification factor (as defined). For positive lift, pitch damping, and static pitching moment, C_{L_α} is positive, C_{m_q} and C_{m_α} negative, so that the exponential term in (26) consists of 'e' to a positive power, and this is, of course, greater than 1.

It will usually be the case, however, that the exponent of 'e' will be fairly small - a value of 0·1 to 0·2 would be typical of current design practice - so that the contribution of the exponential term in (26) will frequently not be very important.

The main effect will generally come from the remaining term in equation (26), namely $\sqrt{2\pi \dfrac{P}{K}\left(\dfrac{B}{A} - 1\right)}$. In current designs this tends to be of the order of 30 to 40. This means, therefore, that if such a re-entry missile is asymmetric to the extent of 1° angular difference between the aerodynamic and inertial axes and it has no precession, the effect on the incidence in the later re-entry stages is the same as if the missile were symmetric but had a re-entry incidence in the form of a precession of 30° to 40° semi-angle round the flight path.

5   NUMERICAL CHECK ON THEORETICAL RESULTS

It was considered desirable to check the validity of the foregoing theoretical analysis for one or two particular cases. For this purpose, a digital calculating machine was programmed to solve the basic differential equation, (4). There is one minor difficulty which had to be overcome - not in the programming as such, but in feeding in appropriate initial conditions. The mathematical solution corresponds to initial conditions

$$\alpha - i\beta = \overline{\dot{\alpha} - i\beta} = 0 \quad \text{at} \quad t = -\infty.$$

To overcome this, values for $\alpha - i\beta$, $\overline{\dot{\alpha} - i\beta}$ were calculated for a later time, appropriate to the very early stages of re-entry, from the expression for s (or $\bar{s}$), and the digital calculation proceeded from there.

Fig.1 shows the digital machine output for three particular cases, with progressively increasing damping. For each of them

$$P = 20, \quad K = 1, \quad A/B = 0 .$$

The damping corresponds to

$$\frac{a_o}{\omega_o^2} = \frac{b_o}{\omega_o^2} = 0 , \quad \frac{a_o}{\omega_o^2} = -\frac{b_o}{\omega_o^2} = \tfrac{1}{4} , \quad \frac{a_o}{\omega_o^2} = -\frac{b_o}{\omega_o^2} = \tfrac{1}{2} ,$$

the lift coefficient being zero in each case.

The first one is the case of zero damping. The other two are equivalent to rather more damping than current re-entry designs normally possess, but it was desired to accentuate the effect.

The mathematical solution, as per equations (23), (24), was checked against the digital machine results. The agreement is virtually perfect for ω/P greater than 1·5, and is still very good for ω/P as low as 1·3.

Other cases were subjected to the same check, with different values of A/B and damping and lift. They present the same general appearance, and confirmation of the mathematical theory.

6   CONCLUSION

A mathematical solution, in convenient form, has been derived for the effect on the angular disturbance of a spinning re-entry missile, which will result from its having a small amount of asymmetry.

The solution is readily applicable in practice, since in the post-resonance region on which interest usually centres, the effect is simply related to the case of the corresponding symmetrical missile.

NOTATION

A,B	moments of inertia in roll and pitch (or yaw)
a,b	parameters defining damping and lift, Section 2
c	parameter relating, inter alia, damping, lift and frequency
K	parameter defining variation of air density with time, $\rho = \rho_o e^{Kt}$
ℓ	aerodynamic reference length
m	mass
P	spin rate
q,r	pitch, yaw angular rates
S	aerodynamic reference area
$s, \bar{s}$ $\left.\begin{array}{c} \\ \\ \end{array}\right\}$ $S, \bar{S}$	Lommel functions (usually with two suffixes)
T	$e^{\frac{Kt}{2}}$
t	time
V	re-entry velocity
$C_{L_\alpha}, C_{m_\alpha}, C_{m_q}$	aerodynamic derivatives
α, β	missile incidence components
γ, δ	trajectory deflection angular components
ε	missile asymmetry
λ	$\dfrac{P}{K}$
μ	$1 + i(2\lambda - \nu)$
ν	$\dfrac{AP}{BK}$

NOTATION (CONTD)

ρ   air density

ω   parameter defining frequency, Section 2

Suffix

o   condition at t = 0

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Attached:

Fig.1 (WE R.3787)
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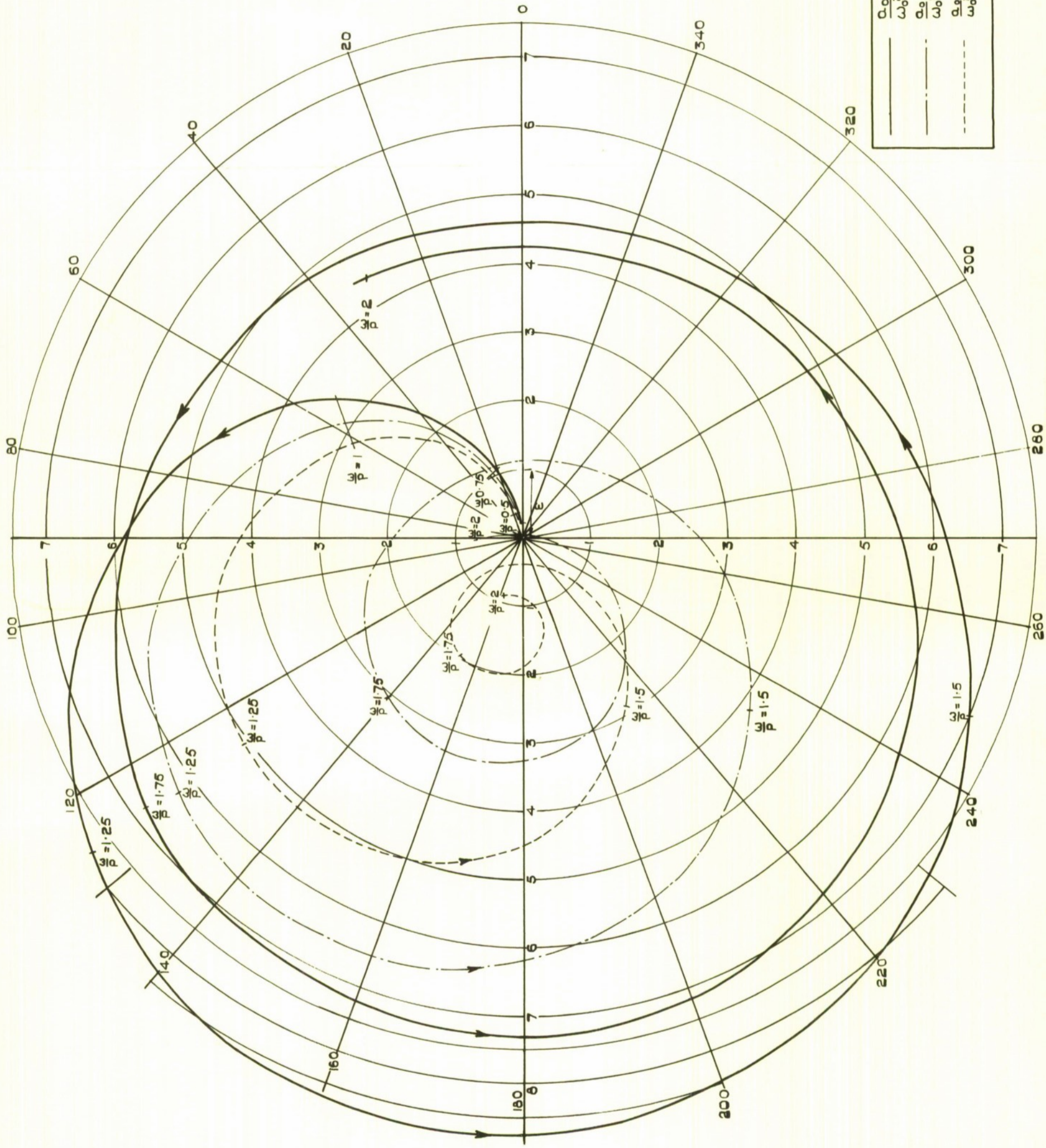Mr. A.K. Weaver, Royal Military College of Science, Shrivenham

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FIG.I. RESPONSE OF SPINNING, ASYMMETRIC MISSILE, DURING RE-ENTRY, REFERRED TO MISSILE AXES

ASYMMETRY, $\varepsilon = 1$ RADIAL UNIT $p = 20$, $k = 1$, $A/B = 0$.

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